THE

MATHEMATICAL GAZETTE.

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The Mathematical Association.

THE Annual Meeting* was held at 2 p.m. on Wednesday, Jan. 5, 1916, at the London Day Training College, Southampton Row, London, W.C., the President, Prof. A. N. Whitehead, F.R.S., in the Chair.

The President called upon Mr. Pendlebury to read the following Report, which was approved:

REPORT OF THE COUNCIL FOR 1915.

During the year 1915, 28 new members have been elected, and the number of members now on the Roll is 723. Of these 9 are honorary members, 37 are life members by composition, 44 are life members under the old rule, and 633 are ordinary members. The number of associates is about 180.

The Council regret to have to record the deaths of Professor F. R. Barrell of Bristol University, Mr. H. Crabtree of Charterhouse, Mr. B. M. R. Denny of Denstone, Mr. G. H. Eyre, Professor W. H. H. Hudson, Mr. F. E. Kitchener, Professor H. W. Lloyd Tanner, Captain J. R. Pound of Shrewsbury, and Mr. T. Wilson. Mr. Kitchener was one of the ten remaining original members of the old A.I.G.T. which was founded in 1871. Professor Barrell was a very valuable member of the Teaching Committee. For several years Professor Lloyd Tanner was a useful co-operator in the editing of the *Gazette*. Captain J. R. Pound and Mr. B. M. R. Denny died in the service of the country.

A report of the "Other Secondary Schools" Committee was issued in the spring of the year, and reports of the Public Schools

*Many members and associates gladly availed themselves of a cordial invitation to attend the meetings of the Public Schools Science Masters' Association on the mornings of Jan. 4th and 5th.

Committee and of the Girls' Schools Committee will be published in an early number of the *Gazette*.

The term for which the Teaching Committees were elected in 1914 will expire immediately. As the work of the Committees has not been free from interruption since the beginning of the war, the Council recommend that the regulations which govern the elections of the Committees be temporarily suspended, and that the term of office of the existing Committees be extended for one year.

Professor R. W. Genese of the University College of Wales, Aberystwyth, whose membership of the Association goes back as far as 1878, took a very active part in the work of the Association in the early days, when the text-book of Plane Geometry and the Syllabuses of Modern Geometry, Solid Geometry and Mechanics were prepared. The Council propose that he be elected a Vice-President.

Two members of the Council, Mr. C. S. Jackson and Professor T. P. Nunn, D.Sc., retire now in rotation and are not eligible for re-election this year. The members present at the Annual Meeting will be asked to nominate and elect their successors.

Mr. H. D. Ellis, who has served the Association as one of the Honorary Secretaries for the last fifteen years, wishes to retire from office. The Council regret that an official connection which has lasted so long should now come to an end, and desire to express their cordial thanks to Mr. Ellis for his valuable services, and their very hearty appreciation of the great benefits which the Association has derived from them.

The Council again desire to acknowledge the indebtedness of the Association to Mr. Greenstreet for his services as editor of the *Mathematical Gazette*: and to offer their thanks to the authorities of the London Day Training College for their kindness in affording accommodation for meetings.

The Treasurer's Report for 1915 was then read and approved.

As the term for which the Teaching Committees were elected in 1914 expires in 1916, it was proposed and carried that the regulations governing the elections should be temporarily suspended, and that the term of office of the Committees should be extended for one year.

Miss M. R. Baldwin, Head Mistress of the High School for Girls, Ludlow, Shropshire, and Mr. G. Goodwill, of the Borough Polytechnic, S.E., were nominated and elected to serve on the Council in place of Mr. C. S. Jackson and Professor T. P. Nunn, D.Sc.

The following Papers were then read:

PRESIDENTIAL ADDRESS.

By Prof. A. N. Whitehead, F.R.S.

THE AIMS OF EDUCATION—A PLEA FOR REFORM.

When I had the honour of being made President of the Mathematical Association, I did not foresee the unusual responsibility which it entailed. It was my intention to take as the theme of a presidential address the consideration of some aspect of those special subjects to which my own researches have principally been directed. Events have forced me to abandon that intention. It is useless to discuss abstract questions in the midst of dominant practical preoccupation. We cannot disregard the present crisis in European civilization. It affects every function of life. In the harder struggle for existence which lies before the nation, all departments of national effort will be reviewed for judgment. The mere necessity for economy in resources will provoke this reformation.

We are concerned with education. This Association, so rich in its membership of educationalists, with the conception of reform as the very reason of its being, is among those bodies which must take the lead in guiding that educational reconstruction which by a sociological law follows every social revolution. We do not want impracticable ideals, only to be realized beyond the clouds in

> "Some wild, weird clime,
> Out of Space, and out of Time."

We require to know what is possible now in England, a nation conscious of its high achievements, and of great failures, shaken to its foundations, distrustful of the old ways, and dreading fantastic novelties.

I will take my courage in both hands, and put before you an outline of educational principles. What I am going to say is of course entirely without your authority, and does not pledge or prejudge any action of the Association. We are primarily concerned only with the intellectual side of education, and, as mathematicians, are naturally concerned to illustrate details more particularly by reference to mathematics. Thus much to explain deliberate omissions in what follows.

Consider now the general and special education of two types of boys, namely those in secondary schools who in after life must form the professional and directing classes in commerce, industry, and public administration, and again those in junior technical schools, and later in advanced continuation classes, who are going to form the class of skilled artisans and foremen of workshops. These two sets compose the educated strength of the nation. We must form no ideals which include less than these entire classes within their scope. What I shall say, will in phraseology apply

more directly to the secondary schools, but with unessential changes it will apply equally to the other group.

What is the first commandment to be obeyed in any educational theme? It is this: Do not teach too many subjects. The second command is this: What you teach, teach thoroughly. The devil in the scholastic world has assumed the form of a general education consisting of scraps of a large number of disconnected subjects; and, with the artfulness of the serpent, he has entrenched himself behind the matriculation examination of the University of London, with a wire entanglement formed by the Oxford and Cambridge schools' examination.

Culture is activity of thought, and receptiveness to beauty, and humane feeling. Scraps of information have nothing to do with it. A merely well-informed man is the most useless bore on God's earth. What we should aim at producing is men who possess both culture and expert knowledge in some special direction. Their expert knowledge will give them the ground to start from, and their culture will lead them as deep as philosophy and as high as art. We have to remember that the valuable intellectual development is self-development, and that it mostly takes place between the ages of 16 and 30. As to training, the most important part is given by mothers before the age of 12. A saying due to Archbishop Temple illustrates my meaning. Surprise was expressed at the success in after-life of a man, who as a boy at Rugby had been somewhat undistinguished. He answered, "It is not what they are at eighteen, it is what they become afterwards that matters."

In training a child to activity of thought, above all things we must beware of what I will call "inert ideas"—that is to say, ideas that are merely received into the mind without being utilized, or tested, or thrown into fresh combinations.

In the history of education, the most striking phenomenon is that schools of learning, which at one epoch are alive with a ferment of genius, in a succeeding generation exhibit merely pedantry and routine. The reason is, that they are overladen with inert ideas. Education with inert ideas is not only useless: it is, above all things, harmful—*Corruptio optimi, pessima.* Except at rare intervals of intellectual ferment, education in the past has been radically infected with inert ideas. That is the reason why uneducated clever women, who have seen much of the world, are in middle life so much the most cultured part of the community. They have been saved from this horrible burden of inert ideas. Every intellectual revolution which has ever stirred humanity into greatness has been a passionate protest against inert ideas. Then, alas, with pathetic ignorance of human psychology, it has proceeded by some educational scheme to bind humanity afresh with inert ideas of its own fashioning.

Let us now ask how in our system of education we are to guard against this mental dry rot. We recur to our two educational commandments, "Do not teach too many subjects," and again, "What you teach, teach thoroughly."

The result of teaching small parts of a large number of subjects is the passive reception of disconnected ideas, not illumined with any spark of vitality. Let the main ideas which are introduced into a child's education be few and important, and let them be thrown into every combination possible. The child should make them his own, and should understand their application here and now in the circumstances of his actual life. From the very beginning of his education, the child should experience the joy of discovery. The discovery which he has to make, is that general ideas give an understanding of that stream of events which pours through his life, which is his life. By understanding I mean more than a mere logical analysis, though that is included. I mean "understanding" in the sense in which it is used in the French proverb, "To understand all, is to forgive all." Pedants sneer at an education which is useful. But if education is not useful, what is it? Is it a talent, to be hidden away in a napkin? Of course, education should be useful, whatever your aim in life. It was useful to Saint Augustine, and it was useful to Napoleon. It is useful, because understanding is useful.

I pass lightly over that understanding which should be given by the literary side of education. It is not peculiarly the function of this Association to consider it. Nor do I wish to be supposed to pronounce on the relative merits of a classical or a modern curriculum. I would only remark that the understanding which we want is an understanding of an insistent present. The only use of a knowledge of the past is to equip us for the present. No more deadly harm can be done to young minds than by depreciation of the present. The present contains all that there is. It is holy ground; for it is the past, and it is the future. At the same time it must be observed that an age is no less past if it existed two hundred years ago than if it existed two thousand years ago. Do not be deceived by the pedantry of dates. The ages of Shakespeare and of Molière are no less past than are the ages of Sophocles and of Vergil. The communion of saints is a great and inspiring assemblage, but it has only one possible hall of meeting, and that is, the present; and the mere lapse of time through which any particular group of saints must travel to reach that meeting-place, makes very little difference.

Passing now to the scientific and logical side of education, we remember that here also ideas which are not utilized are positively harmful. By utilizing an idea, I mean relating it to that stream, compounded of sense perceptions, feelings, hopes, desires, and of mental activities relating thought to thought, which forms our life. I can imagine a set of beings which might fortify their souls

by passively reviewing disconnected ideas. Humanity is not built that way—except perhaps some editors of newspapers.

In scientific training, the first thing to do with an idea is to prove it. But allow me for one moment to extend the meaning of 'prove'; I mean—to prove its worth. Now an idea is not worth much unless the propositions in which it is embodied are true. Accordingly an essential part of the proof of an idea is the proof, either by experiment or by logic, of the truth of the propositions. But it is not essential that this proof of the truth should constitute the first introduction to the idea. After all, its assertion by the authority of respectable teachers is sufficient evidence to begin with. In our first contact with a set of propositions, we commence by appreciating their importance. That is what we all do in after-life. We do not attempt, in the strict sense, to prove or to disprove anything, unless its importance makes it worthy of that honour. These two processes of proof, in the narrow sense, and of appreciation do not require a rigid separation in time. Both can be proceeded with nearly concurrently. But in so far as either process must have the priority, it should be that of appreciation by use.

Furthermore, we should not endeavour to use propositions in isolation. Emphatically I do not mean, a neat little set of experiments to illustrate proposition I. and then the proof of proposition I., a neat little set of experiments to illustrate proposition II. and then the proof of Proposition II., and so on to the end of the book. Nothing could be more boring. Inter-related truths are utilized *en bloc*, and the various propositions are employed in any order, and with any reiteration. Choose some important applications of your theoretical subject; and study them concurrently with the systematic theoretical exposition. Keep the theoretical exposition short and simple, but let it be strict and rigid so far as it goes. It should not be too long for it easily to be known with thoroughness and accuracy. The consequences of a plethora of half-digested theoretical knowledge are deplorable. Also the theory should not be muddled up with the practice. The child should have no doubt when it is proving and when it is utilizing. My point is that what is proved should be utilized, and that what is utilized should—so far as is practicable—be proved. I am far from asserting that proof and utilization are the same thing.

At this point of my discourse, I can most directly carry forward my argument in the outward form of a digression. We are only just realizing that the art and science of education require a genius and a study of their own; and that this genius and this science are more than a bare knowledge of some branch of science or of literature. This truth was partially perceived in the past generation; and headmasters, somewhat crudely, were apt to supersede learning in their colleagues by requiring left-hand bowling and a taste for

football. But culture is more than cricket, and more than football, and more than extent of knowledge.

Education is the acquisition of the art of the utilization of knowledge. This is an art very difficult to impart. Whenever a text-book is written of real educational worth, you may be quite certain that some reviewer will say that it will be difficult to teach from it. Of course, it will be difficult to teach from it. If it were easy, the book ought to be burned; for it cannot be educational. In education, as elsewhere, the broad primrose path leads to a nasty place. This evil path is represented by a book or a set of lectures which will practically enable the student to learn by heart all the questions likely to be asked at the next external examination. And I may say in passing that no educational system is possible unless every question directly asked of a pupil at any examination is either framed or modified by the actual teacher of that pupil in that subject. The external assessor may report on the curriculum or on the performance of the pupils, but never should be allowed to ask the pupil a question which has not been strictly supervised by the actual teacher, or at least inspired by a long conference with him. There are a few exceptions to this rule, but they are exceptions, and could easily be allowed for under the general rule.

We now return to my previous point, that theoretical ideas should always find important applications within the pupil's curriculum. This is not an easy doctrine to apply, but a very hard one. It contains within itself the problem of keeping knowledge alive, of preventing it from becoming inert, which is the central problem of all education.

The best procedure will depend on several factors, none of which can be neglected, namely, the genius of the teacher, the intellectual type of the pupils, their prospects in life, the opportunities offered by the immediate surroundings of the school, and allied factors of this sort. It is for this reason that the uniform external examination is so deadly. We do not denounce it because we are cranks, and like denouncing established things. We are not so childish. Also, of course, such examinations have their use in testing slackness. Our reason of dislike is very definite and very practical. It kills the best part of culture. When you analyse in the light of experience the central task of education, you find that its successful accomplishment depends on a delicate adjustment of many variable factors. The reason is that we are dealing with human minds, and not with dead matter. The evocation of curiosity, of judgment, of the power of mastering a complicated tangle of circumstances, the use of theory in giving foresight in special cases—all these powers are not to be imparted by a set rule embodied in one schedule of examination subjects.

I appeal to you, as practical teachers. With good discipline, it is

always possible to pump into the minds of a class a certain quantity of inert knowledge. You take a text-book and make them learn it. So far, so good. The child then knows how to solve a quadratic equation. But what is the point of teaching a child to solve a quadratic equation ? There is a traditional answer to this question. It runs thus : The mind is an instrument, you first sharpen it, and then use it ; the acquisition of the power of solving a quadratic equation is part of the process of sharpening the mind. Now there is just enough truth in this answer to have made it live through the ages. But for all its half-truth, it embodies a radical error which bids fair to stifle the genius of the modern world. I do not know who was first responsible for this analogy of the mind to a dead instrument. For aught I know, it may have been one of the seven wise men of Greece, or a committee of the whole lot of them. Whoever was the originator, there can be no doubt of the authority which it has acquired by the continuous approval which it has received from eminent persons. But whatever its weight of authority, whatever the high approval which it can quote, I have no hesitation in denouncing it as one of the most fatal, erroneous, and dangerous conceptions ever introduced into the theory of education. The mind is never passive ; it is a perpetual activity, delicate, receptive, responsive to stimulus. You cannot postpone its life until you have sharpened it. Whatever interest attaches to your subject-matter, must be evoked here and now ; whatever powers you are strengthening in the pupil, must be exercised here and now ; whatever possibilities of mental life your teaching should impart, must be exhibited here and now. That is the golden rule of education, and a very difficult rule to follow.

The difficulty is just this : the apprehension of general ideas, intellectual habits of mind, and pleasurable interest in mental achievement can be evoked by no form of words, however accurately adjusted. All practical teachers know that education is a patient process of the mastery of details, minute by minute, hour by hour, day by day. There is no royal road to learning through an airy path of brilliant generalizations. There is a proverb about the difficulty of seeing the wood because of the trees. That difficulty is exactly the point which I am enforcing. The problem of education is to make the pupil see the wood by means of the trees.

The solution which I am urging, is to eradicate the fatal disconnection of subjects which kills the vitality of our modern curriculum. There is only one subject matter for education, and that is Life in all its manifestations. Instead of this single unity, we offer children—Algebra, from which nothing follows ; Geometry, from which nothing follows ; Science, from which nothing follows ; History, from which nothing follows ; a Couple of Languages, never mastered ; and lastly, most dreary of all, Literature, represented by plays of Shakespeare, with philological notes and short

analyses of plot and character to be in substance committed to memory. Can such a list be said to represent Life, as it is known in the midst of the living of it? The best that can be said of it, is that it is a rapid table of contents which a deity might run over in his mind while he was thinking of creating a world, and had not yet determined how to put it together.

Let us now return to quadratic equations. We still have on hand the unanswered question, Why should children be taught their solution? Unless quadratic equations fit into a connected curriculum, of course there is no reason to teach anything about them. Furthermore, extensive as should be the place of mathematics in a complete culture, I am a little doubtful whether for many types of boys algebraic solutions of quadratic equations do not lie on the specialist side of mathematics. I may here remind you that as yet I have not said anything of the psychology or the content of the specialism, which is so necessary a part of an ideal education. But all that is an evasion of our real question, and I merely state it in order to avoid being misunderstood in my answer.

Quadratic equations are part of algebra, and algebra is the intellectual instrument which has been created for rendering clear the quantitative aspects of the world. There is no getting out of it. Through and through the world is infected with quantity. To talk sense, is to talk in quantities. It is no use saying that the nation is large,—How large? It is no use saying that radium is scarce,—How scarce? You cannot evade quantity. You may fly to poetry and to music, and quantity and number will face you in your rhythms and your octaves. Elegant intellects which despise the theory of quantity, are but half developed. They are more to be pitied than blamed. The scraps of gibberish which in their schooldays were taught to them in the name of algebra, deserve some contempt.

This question of the degeneration of algebra into gibberish, both in word and in fact, affords a pathetic instance of the uselessness of reforming educational schedules without a clear conception of the attributes which you wish to evoke in the living minds of the children. A few years ago there was an outcry that school algebra was in need of reform, but there was a general agreement that graphs would put everything right. So all sorts of things were extruded, and graphs were introduced. So far as I can see with no sort of idea behind them, but just graphs. Now every examination paper has one or two questions on graphs. Personally, I am an enthusiastic adherent of graphs. But I wonder whether as yet we have gained very much. You cannot put life into any schedule of general education unless you succeed in exhibiting its relation to some essential characteristic of all intelligent or emotional perception. It is a hard saying, but it is true; and I do not see how to make it any easier. In making these little formal altera-

tions you are beaten by the very nature of things. You are pitted against too skilful an adversary, who will see to it that the pea is always under the other thimble.

Reformation must begin at the other end. First, you must make up your mind as to those quantitative aspects of the world which are simple enough to be introduced into general education; then a schedule of algebra should be framed which will about find its exemplification in these applications. We need not fear for our pet graphs, they will be there in plenty when we once begin to treat algebra as a serious means of studying the world. Some of the simplest applications will be found in the quantities which occur in the simplest study of society. The curves of history are more vivid and more informing than the dry catalogues of names and dates which comprise the greater part of that arid school study. What purpose is effected by a catalogue of undistinguished kings and queens? Tom, Dick, or Harry, they are all dead. General resurrections are failures, and are better postponed. The quantitative flux of the forces of modern society are capable of very simple exhibition. Meanwhile the idea of the variable, of the function, of rate of change, of equations and their solution, of elimination, are being studied as an abstract science for their own sake. Not of course in the pompous phrases with which I am alluding to them here, but with that iteration of simple special cases proper to teaching.

If this course be followed, the route from Chaucer to the Black Death, from the Black Death to modern Labour troubles, will connect the tales of the medieval pilgrims with the abstract science of algebra, both yielding diverse aspects of that single theme, Life. I know what most of you are thinking at this point. It is that the exact course which I have sketched out is not the particular one which you would have chosen, or even see how to work. I quite agree. I am not claiming that I could do it myself. But your objection is the precise reason why a common external examination system is fatal to education. The process of exhibiting the applications of knowledge must, for its success, essentially depend on the character of the pupils and the genius of the teacher. Of course I have left out the easiest applications with which most of us are more at home. I mean the quantitative sides of sciences, such as mechanics and physics.

My meaning can be illustrated by looking more closely into a special case of this type of application. In my rough catalogue of the sort of subjects which should form the schedule for Algebra, I mentioned Elimination. It was not put there by accident, for it covers a very important body of thought.

In the first place, there is the abstract process of algebraic elimination for suitable simple cases. The pupil acquires a firm grasp of this by the process, inevitable in education, of working an adequate

number of examples. Again, there are the graphical solutions of the same problem. Then we consider the significance in the external world. We consider the velocity, time, space, acceleration diagrams. We take uniform acceleration; we eliminate "t" between

$$v = u + ft, \text{ and } s = ut + \tfrac{1}{2}ft^2,$$

and eliminate "s" between

$$v^2 = u^2 + 2fs, \text{ and } s = ut + \tfrac{1}{2}ft^2.$$

Then we remember that constant acceleration is a very special case, and we consider graphical solutions for empirically given variations of v or of f. In preference, we use those empirical formulae which occur in the pupil's experimental work. We compare the strong and weak points of the algebraic and graphical solutions.

Again, in the same connection we plot the statistics of social phenomena against the time. We then eliminate the time between suitable pairs. We can speculate how far we have exhibited a real causal connection, or how far a mere temporal coincidence. We notice that we might have plotted against the time one set of statistics for one country and another set for another country, and thus, with suitable choice of subjects, have obtained graphs which certainly exhibited mere coincidence. Also other graphs exhibit obvious causal connections. We wonder how to discriminate. And so are drawn on as far as we will.

But in considering this description, I must beg you to remember what I have been insisting on above. In the first place, one train of thought will not suit all groups of children. For example, I should expect that artisan children will want something more concrete and, in a sense, swifter than I have set down here. Perhaps I am wrong, but that is what I should guess. In the second place, I am not contemplating one beautiful lecture stimulating, once and for all, an admiring class. That is not the way in which education proceeds. No; all the time the pupils are hard at work solving examples, drawing graphs, and making experiments, until they have a thorough hold on the whole subject. I am describing the interspersed explanations, the directions which should be given to their thoughts. The pupils have got to be made to feel that they are studying something, and are not merely executing intellectual minuets.

In this connection the excellence of some of the most recent text-books on elementary algebra emanating from members of this Association, should create an epoch in the teaching of the subject.

Finally, if you are teaching pupils for some general examination, the problem of sound teaching is greatly complicated. Have you ever noticed the zig-zag moulding round a Norman arch? The ancient work is beautiful, the modern work is hideous. The reason is that the modern work is done to exact measure, the ancient work is varied according to the idiosyncrasy of the workman.

Here it is crowded, and there it is expanded. Now the essence of getting pupils through examinations is to give equal weight to all parts of the schedule. But mankind is naturally specialist. One man sees a whole subject, where another can find only a few detached examples. I know that it seems contradictory to allow for specialism in a curriculum especially designed for a broad culture. Without contradictions the world would be simpler, and perhaps duller. But I am certain that in education wherever you exclude specialism you destroy life.

We now come to the other great branch of a general mathematical education, namely Geometry. The same principles apply. The theoretical part should be clear cut, rigid, short, and important. Every proposition not absolutely necessary to exhibit the main connection of ideas should be cut out, but the great fundamental ideas should all be there. No omission of concepts, such as those of Similarity and Proportion. We must remember that, owing to the aid rendered by the visual presence of a figure, Geometry is a field of unequalled excellence for the exercise of the deductive faculties of reasoning. Then of course there follows Geometrical Drawing, with its training for the hand and eye.

But, like Algebra, Geometry and Geometrical Drawing must be extended beyond the mere circle of geometrical ideas. In an industrial neighbourhood, machinery and workshop practice form the appropriate extension. For example, in the London Polytechnics this has been achieved with conspicuous success. For many secondary schools I suggest that surveying and maps are the natural applications. In particular, plane-table surveying should lead pupils to a vivid apprehension of the immediate application of geometric truths. Simple drawing apparatus, a surveyor's chain, and a prismatic compass, should enable the pupils to rise from the survey and mensuration of a field to the construction of the map of a small district. The best education is to be found in gaining the utmost information from the simplest apparatus. The provision of elaborate instruments is greatly to be deprecated. To have constructed the map of a small district, to have considered its roads, its contours, its geology, its climate, its relation to other districts, the effects on the status of its inhabitants, will teach more history and geography than any knowledge of Perkin Warbeck or of Behren's Straits. I mean not a nebulous lecture on the subject, but a serious investigation in which the real facts are definitely ascertained by the aid of accurate theroetical knowledge. A typical mathematical problem should be: Survey such and such a field, draw a plan of it to such and such a scale, and on the same linear scale construct a square of equal area. It would be quite a good procedure to impart the necessary geometrical propositions without their proofs. Then, concurrently in the same term, the proofs of the propositions would be learnt while the survey was being made.

Fortunately, the specialist side of education presents an easier problem than does the provision of a general culture. For this there are many reasons. One is that many of the principles of procedure to be observed are the same in both cases, and it is unnecessary to recapitulate. Another reason is that specialist training takes place—or should take place—at a more advanced stage of the pupil's course, and thus there is easier material to work upon. But undoubtedly the chief reason is that the specialist study is normally a study of peculiar interest to the student. He is studying it because, for some reason, he wants to know it. This makes all the difference. The general culture is designed to foster an activity of mind; the specialist course utilizes this activity. But it does not do to lay too much stress on these neat antitheses. As we have already seen, in the general course foci of special interest will arise; and similarly in the special study, the external connections of the subject drag thought outwards.

Again, there is not one course of study which merely gives general culture, and another which gives special knowledge. The subjects pursued for the sake of a general education are special subjects specially studied; and, on the other hand, one of the ways of encouraging general mental activity is to foster a special devotion. You may not divide the seamless coat of learning. What education has to impart is an intimate sense for the power of ideas, for the beauty of ideas, and for the structure of ideas, together with a particular body of knowledge which has peculiar reference to the life of the being possessing it.

The appreciation of the structure of ideas is that side of a cultured mind which can only grow under the influence of a special study. I mean that eye for the whole chess-board, for the bearing of one set of ideas on another. Nothing but a special study can give any appreciation for the exact formulation of general ideas, for their relations when formulated, for their service in the comprehension of life. A mind so disciplined should be both more abstract and more concrete. It has been trained in the comprehension of abstract thought and in the analysis of facts.

Finally, there should grow the most austere of all mental qualities; I mean the sense for style. It is an aesthetic sense, based on admiration for the direct attainment of a foreseen end, simply and without waste. Style in art, style in literature, style in science, style in logic, style in practical execution have fundamentally the same aesthetic qualities, namely, attainment and restraint. The love of a subject in itself and for itself, where it is not the sleepy pleasure of pacing a mental quarter-deck, is the love of style as manifested in that study.

Here we are brought back to the position from which we started, the utility of education. Style, in its finest sense, is the last acquirement of the educated mind; it is also the most useful. It pervades the whole being. The administrator with a sense for style, hates

waste; the engineer with a sense for style, economises his material; the artisan with a sense for style, prefers good work. Style is the ultimate morality of mind.

But above style, and above knowledge, there is something, a vague shape like fate above the Greek gods. That something is Power. Style is the fashioning of power, the restraining of power. But, after all, the power of attainment of the desired end is fundamental. The first thing is to get there. Do not bother about your style, but solve your problem, justifying the ways of God to man, administer your province, or do whatever else is set before you.

Where then does style help? In this, with style the end is attained without side issues, without raising undesirable inflammations. With style you attain your end and nothing but your end. With style the effect of your activity is calculable, and foresight is the last gift of gods to men. With style your power is increased, for your mind is not distracted with irrelevancies, and you are more likely to attain your object. Now style is the exclusive privilege of the expert. Whoever heard of the style of an amateur painter, of the style of an amateur poet? Style is always the product of specialist study, the peculiar contribution of specialism to culture.

English education in its present phase suffers from a lack of definite aim, and from an external machinery which kills its vitality. Hitherto in this address I have been considering the aims which should govern education. In this respect England halts between two opinions. It has not decided whether to produce amateurs or experts. The profound change in the world which the nineteenth century has produced is that the growth of knowledge has given foresight. The amateur is essentially a man with appreciation and with immense versatility in mastering a given routine. But he lacks the foresight which comes from special knowledge. The object of this address is to suggest how to produce the expert without loss of the essential virtues of the amateur. The machinery of our secondary education is rigid where it should be yielding, and lax where it should be rigid. Every school is bound on pain of extinction to train its boys for a small set of definite examinations. No headmaster has a free hand to develop his general education or his specialist studies in accordance with the opportunities of his school, which are created by its staff, its environment, its class of boys, and its endowments. I suggest that no system of external tests which aims primarily at examining individual scholars can result in anything but educational waste.

Primarily it is the schools and not the scholars which should be inspected. Each school should grant its own leaving certificates, based on its own curriculum. The standards of these schools should be sampled and corrected. But the first requisite for educational

reform is the school as a unit, with its approved curriculum based on its own needs, and evolved by its own staff. If we fail to secure that, we simply fall from one formalism into another, from one dunghill of inert ideas into another.

In stating that the school is the true educational unit in any national system for the safeguarding of efficiency, I have conceived the alternative system as being the external examination of the individual scholar. But every Scylla is faced by its Charybdis—or, in more homely language, there is a ditch on both sides of the road. It will be equally fatal to education if we fall into the hands of a supervising department which is under the impression that it can divide all schools into two or three rigid categories, each type being forced to adopt a rigid curriculum. When I say that the school is the educational unit, I mean exactly what I say, no larger unit, no smaller unit. Each school must have the claim to be considered in relation to its special circumstances. The classifying of schools for some purposes is necessary. But no absolutely rigid curriculum, not modified by its own staff, should be permissible. Exactly the same principles apply, with the proper modifications, to universities and to technical colleges.

When one considers in its length and in its breadth the importance of this question of the education of a nation's young, the broken lives, the defeated hopes, the national failures, which result from the frivolous inertia with which it is treated, it is difficult to restrain within oneself a savage rage. In the conditions of modern life the rule is absolute, the race which does not value trained intelligence is doomed. Not all your heroism, not all your social charm, not all your wit, not all your victories on land or at sea, can move back the finger of fate. To-day we maintain ourselves. To-morrow science will have moved forward yet one more step, and there will be no appeal from the judgment which will then be pronounced on the uneducated.

We can be content with no less than the old summary of educational ideal which has been current at any time from the dawn of our civilization. The essence of education is that it be religious.

Pray, what is religious education?

A religious education is an education which inculcates duty and reverence. Duty arises from our potential control over the course of events. Where attainable knowledge could have changed the issue, ignorance has the guilt of vice. And the foundation of reverence is this perception, that the present holds within itself the complete sum of existence, backwards and forwards, that whole amplitude of time, which is eternity.

A vote of thanks to the President for his Address was proposed by Mr. A. Lodge and carried unanimously.

ACCURACY IN ARITHMETIC.

By G. W. Palmer, M.A.

During the past two years the Public Schools Special Committee has been considering a report on the teaching of Arithmetic. One of the chief principles underlying the report may be put shortly as follows. The Arithmetic course in the past has contained a good deal that a very large number of teachers regard as unprofitable, or at any rate as less profitable than other subjects that might be substituted for it. It is proposed that such parts of the Arithmetic course should not be set in the compulsory Arithmetic paper in Pass Examinations, so that those people who do not wish to teach them are not compelled to do so.

It is also suggested that the time saved by omitting this work should be used *partly* in obtaining a higher standard of accuracy in straightforward computation in Arithmetic, *partly* in pushing on to other subjects, such as Trigonometry and Mechanics.

It is with the former of these suggestions that I am concerned in this short paper, viz.: that by restricting the Arithmetic course it is possible to obtain a higher standard of accuracy in straightforward computation.

Of course some people may resent this suggestion. They may urge that teachers have always done their utmost to secure accuracy from their pupils, and that there is nothing *more* to be done. This seems to me to miss the point. If you regard accuracy as one of the most important, perhaps *the* most important, quality of a boy's work in Arithmetic, and if you are to insist on a high standard of accuracy, it is absolutely necessary that the work should be of such a kind that a high standard is possible.

I can well remember learning, in company with some twenty other boys, to reduce sq. ins. to a compound quantity in sq. ft., sq. yd., sq. poles, etc., up to sq. miles. The average age of the form was perhaps twelve. Unless we were to devote to this dreary task an amount of time altogether out of proportion to its value, it would have been ludicrous for the master to have said to us: "Once you have learned this process, it is purely mechanical. You know your multiplication table. If you will check your work, you can detect your mistakes. I shall expect 95 per cent. of these sums to be right." Probably the master was well content if we got as many right as wrong. We were taught to work carefully, but, of course, the work was much too heavy for us. Because "Compound Quantities" or "Weights and Measures" formed Chapter II. of the Arithmetic, it fell to our lot to struggle with this difficult task at such an early age that accuracy was out of the question.

But, on the other hand, if you are teaching Multiplication and Division of Decimals, and confine the work to numbers of not more than three figures; if you teach the use of rough checks, and insist that multiplication should be checked by division and *vice versa*, then it may reasonably be argued that a high standard of accuracy is possible.

Of course the attitude of the boy is different in the two cases. In the old days he was taught to work *carefully* line by line, but as soon as he reached the answer he had in the great majority of cases no further interest in the work and passed on. He rarely paused to consider whether the answer was possible; if he noticed that it did not seem reasonable, he probably thought "Kismet—Fortune had been unkind; perhaps in the next question she will smile upon me."

But in the other case the *accuracy of the result* is the all-important

matter. The work is well within the boy's power; checks are therefore feasible; there ought to be more confidence in the result.

During the past two or three years I have had many arguments about the degree of accuracy that is possible in simple straightforward work in Arithmetic. Evidently it is not a question that is likely to be settled by *a priori* reasoning, and it is not surprising that no definite result was reached in that way. On the other hand, it seemed an easy matter to settle by the statistical method. But, though doubtless results obtained in this manner exist, I do not know of any that are generally accessible.

During the past eighteen months, in one of the Public Schools, several special examination papers have been set, which give us a certain amount of evidence of the kind that we require. These papers were originally set to test the standard of accuracy in that particular school and without any reference to such a paper as this, and in consequence the records of some of them are incomplete. Some eight were set altogether, and one set of question papers and two sets of results have been cyclostyled and circulated among the members present.*

These particular sets were chosen, not as the best results, but as the most complete. They chance, however, to be among the best, but the others, with one exception, did not differ very greatly from them.

You will notice that the average age of the boys in the school taken is very considerably less than in most of the Public Schools. On the other hand, the time given to Mathematics is above the average, being between four and six hours a week in school, apart from preparation.

Block A of the Classical Side is roughly parallel to Block A of the Modern, and so for Blocks B and C.

The boys were allowed as much time as they wanted, within reason, for the papers. I have not entered the results of the bottom set in each of the two parts of the school, as they contained a large number of new boys.

In each paper the first question was on the Multiplication and Division of Decimals; for Blocks B and C, of the form $m \times n$ or $m \div n$; for Block A it was rather more complicated. Taking these questions in Paper I. you will see that 39 boys out of 570 made mistakes, *i.e.* about 7 per cent. The number of mistakes in the position of the decimal point in these questions was 6, *i.e.* just over 1 per cent. Of these 6 mistakes, 3 came from one set. On the Modern Side there was only 1 mistake among 267 boys.

Though in the bulk, the evidence of one paper was much the same as another, the separate items varied considerably. *E.g.* in Paper II. the number of mistakes in the decimal point is 14 instead of 6, but it will be found that 9 of these occur in two sets, and that the average for the rest of the school is much the same as before.

The results of Paper II. contain in the last column the number of boys who used the school methods in multiplication and division of decimals. The school methods are, in the lower forms, to multiply or divide by changing to standard form; in the upper forms, by working with significant figures and placing the decimal point by a rough check. As far as these special papers were concerned, masters were instructed to allow the boys to adopt any method they pleased. This in most cases they interpreted to mean that they need say nothing to the boys about method, but one master stated definitely to them that they could employ whatever method was most likely to give an

* These are given on pp. 206-8.

accurate answer, with the result, as you will see, that 28 out of 29 preferred unorthodox methods.

An attempt has since been made to find out how far the accuracy of the result in multiplying and dividing with decimals is affected by forcing boys to use particular methods. In so far as any conclusions could be drawn, they seemed to point to a gain in accuracy in the lower forms by allowing boys to use their favourite methods, but it was not clear that there was any gain in the upper forms. But I do not think that any definite conclusions can be reached until the experiment has been carried out on a more extensive scale.

I need not say much about the rest of the papers, the figures speak for themselves. Of course, the questions were made as straightforward as possible, so that the papers should be a test of accuracy only.

One of the papers brought out what seemed to me a curious fact, though doubtless it is well known to many present. I stated early in my remarks that one paper showed considerably worse results than any of the others. This was due to the fact that one of the questions was what is sometimes called a Long Tot. The younger boys were asked to add 10 or 12 sums of money, the older boys 18 or 20. 93 boys failed to get this question right, though the numbers who failed in the other two questions of the same paper were only 26 and 21. Evidently a "Long Tot" is a searching test of accuracy to the ordinary school-boy.

I should like to add that, though I have made mention of the Report of the Public Schools Special Committee, I am in no way speaking on behalf of that committee, but simply giving my own private opinions in this paper.

I feel very apologetic for having taken up the time of this meeting with the account of so small an experiment. But it seems to me desirable that mathematical teachers should have some idea of the standard of accuracy that may reasonably be expected in quite simple straightforward Arithmetic, in which the numbers involved are not too large. The results given in this paper deal with only one school, and it may consequently be argued that they have little general value. If a similar experiment could be carried out over a wider area, the results ought to be much more valuable, and such a piece of work seems well worth doing.

STATISTICS OF RESULTS OF EXAMINATIONS IN ARITHMETIC TO ILLUSTRATE THE PAPER BY MR. G. W. PALMER.

PAPER I. [QUESTIONS.] (1)

Classical Side.

A. 1. Value of $\frac{13{\cdot}83 \times 0{\cdot}376}{260{\cdot}2}$ to three figures.

2. An express goes from London to Bristol (117¼ mi.) in 2 hr. 13 min. When does it reach Swindon (77¼ mi.)?

3. Value of 3·651 times £2 13s. 10d. to the nearest penny.

B. 1. Multiply 17·83 by 0·428.

2. Simple interest on £563 15s. 0d. for 8 months at 4 % to the nearest penny.

3. Number of pounds in 13 t. 7 cwt. 20 lb.

C. 1. Divide 0·364 by 23·9 to four places.

2. Add the following :—(Six sums of money).

3. Number of pence in £635 17s. 9d.

Modern Side.

A. 1. Value of $\sqrt{\frac{0{\cdot}635}{43{\cdot}72}}$ to three places.

2. An express goes from London to Crewe (158 mi.) in 163 min. When does it pass Rugby (82¼ mi.) ?

3. Value of 6·214 times £1 14s. 4d. to the nearest penny.

B. 1. Multiply 0·325 by 83·61.

2. Simple Interest on £724 14s. 0d. for 5 months at 4 % to the nearest penny.

3. Number of pounds in 22 t. 17 cwt. 24 lb.

C. 1. Divide 0·837 by 37·6 to four places.

2. Add the following :—(Six sums of money).

3. Number of pence in £86 17s. 8d.

PAPER I. [RESULTS.]

Classes.		Number of Boys.	Average Age.	Number of Mistakes.			
				Qu. 1.	Qu. 2.	Qu. 3.	In Decimal Pt. in Qu 1.
Classical Side.							
A.	1.	17	15·10	3	0	4	1
	2.	15	15·7	0	1	5	0
	3.	20	15·5	1	1	1	0
	4.	25	15·3	0	8	1	0
B.	1.	22	14·4	1	1	0	0
	2.	25	13·11	1	2	1	1
	3.	26	14·0	0	3	0	0
	4.	27	13·6	0	2	0	0
	5.	27	13·0	1	11	1	0
	6.	27	13·8	0	0	3	0
	7.	20	12·11	0	5	3	0
C.	1.	26	12·2	2	2	3	0
	2.	26	12·2	4	1	2	3
	3.						
Modern Side.							
A.	1.	23	16·0	1	1	1	0
	2.	23	15·6	8	0	10	0
	3.	22	15·5	3	4	14	0
	4.	25	15·7	5	3	4	0
B.	1.	24	14·6	0	0	0	0
	2.	25	14·9	0	0	0	0
	3.	26	14·8	0	2	0	0
	4.	24	14·8	1	8	2	0
C.	1.	25	13·2	2	1	0	0
	2.	26	13·5	6	0	0	1
	3.	23	13·0	0	0	0	0
	4.						

Number of boys examined - - -	570
Number of mistakes in Qu. 1 - - -	39
,, ,, ,, Qu. 2 - - -	56
,, ,, ,, Qu. 3 - - -	55
,, ,, ,, Dec. Pt. in Qu. 1 -	6

PAPER II. [RESULTS.]

(The questions were very similar to those set in Paper I.)

Classes.		No of Boys.	Average Age.	No. of Mistakes. Qu. 1.	No. of Mistakes. Qu. 2.	No. of Mistakes. Qu. 3.	No. who did not get all Right.	No. of Mistakes in Decimal Pt. in Qu. 1.	Did not use the Recognised Method in Qu. 1.
Classical Side.									
A.	1.	22	15·10	0	1	0	1	0	0
	2.	19	15·6	0	1	1	2	0	2
	3.	20	15·5	2	1	2	4	0	0
	4.	20	15·8	1	1	0	1	0	0
B.	1.	20	14·6	2	2	0	3	0	4
	2.	27	14·0	1	2	2	4	0	0
	3.	27	13·6	1	1	1	3	1	0
	4.	22	13·8	4	4	0	8	1	4
	5.	25	13·2	0	1	0	1	0	0
	6.	25	13·5	4	1	3	7	4	3
	7.	21	12·7	0	0	0	0	0	1
C.	1.	29	12·5	1	0	0	1	0	28
	2.	28	12·1	1	0	2	3	1	0
	3.								
Modern Side.									
A.	1.	22	16·1	0	0	2	2	0	0
	2.	20	15·7	1	2	1	3	1	1
	3.	24	15·4	0	0	2	2	0	10
	4.	18	15·3	1	2	1	3	0	2
B.	1.	14	14·7	1	2	0	3	0	0
	2.	17	14·3	0	1	0	1	0	0
	3.	26	13·9	0	1	1	2	0	5
	4.	26	14·0	3	2	2	7	0	0
C.	1.	23	12·10	5	0	3	8	5	10
	2.	24	13·8	2	0	1	3	1	1
	3.	24	13·2	0	0	1	1	0	0
	4.								

Number of boys examined - - -	543
Number who did not get all the Paper right	73
Number of mistakes in Qu. 1 - - -	30
,, ,, ,, Qu. 2 - - -	25
,, ,, ,, Qu. 3 - - -	25
,, ,, ,, Dec. Pt. in Qu. 1 -	14

AS TO THE COMPLETENESS OF MATHEMATICAL TABLES REQUIRED IN SCHOOL AND EXAMINATION WORK.

By A. Lodge, M.A.

My special object is to consider how extensive the tables should be for school use, and how far pupils who have used fairly complete sets of tables at school are likely to be handicapped in examinations for which scantier tables are provided, or tables less conveniently arranged, and to what extent steps could be taken to minimise this handicap. It is to be hoped that most of the important examining bodies will before long adopt one or other of the completer sets, as it seems a pity to give the candidates needless difficulty over mere computation. I confine my remarks to 4-figure tables. And, first, as to the tables which are the most essential.

(1) *Logarithms.* All the books, of course, have these. The best arrangement seems to be that in which thick lines separate the main columns into groups, thus :

0	1	2	3	4	5	6	7	8	9

and similarly for the difference columns. There is a good deal of eye-trouble saved by this visible grouping. Some tables give an extra wide gap instead of thick lines, and this is as good, but requires a larger page.

(2) *Sines, cosines, tangents, cotangents, secants and cosecants.* There is no doubt that the most convenient tables for beginners are those in which all these functions are separate, six distinct tables. If they are combined in pairs, the complementary functions being read from the foot and right-hand side of the page, many beginners make mistakes in choice of column. I have known many mistakes made by candidates in this way.

The trouble is that if the school tables separate all the functions and the examination tables do not, many of the candidates are practically sure to get into difficulties, either making actual mistakes, or at least losing valuable time. Then as to the argument: should they be given to 10′ intervals or 6′? My own feeling is that 6′ intervals are the best, for two reasons: (1) they are decimal divisions of the degree, (2) the differences are smaller. As against the first advantage is the question of whether the differences should be given to minutes or to hundredths of a degree. My own preference is for minutes, but some good tables give the hundredths, and I see Prof. Huntingdon's tables give the hundredths up to half way: ·01, ·02, ·03, ·04, ·05. Some sets of tables omit secants and cosecants, which seems to me lamentable, as so often the absence of such a table entails needless labour and loss of time.

(3) *Logarithms of the circular functions* (which should not be labelled "logarithmic"). It is easiest for the beginner if there are also six distinct sets of these, but it is by no means so necessary, as it is nearly as easy to use log cos instead of log sec, and so on, subtracting instead of adding, or better still, drill should be given in writing log sec from the log cos table, and similarly for the other reciprocals. Practice in writing down the logarithm of a reciprocal on sight from the logarithm of the number itself is very valuable, and facility is very easily acquired. My own feeling is that all the tables should be provided, but that the pupils should be taught and encouraged to work chiefly with log sin, log cos and log tan, as they become expert.

It is very desirable that tables to single seconds should be provided for at least the first 5 degrees in the case of log sines and log tangents. Prof. Knott in his tables gives such a detailed table up to the 8th degree for log sines, with a marginal table of additions required for the log tans.

(4) *Tables of reciprocals.*
(5) *Tables of squares.*
} These often shorten calculations immensely, and it is here that examination tables are so often deficient.

(6) *Radian tables.* These are of great value, and frequently save a good deal of labour. The most convenient arrangement for quick use is that of Prof. Knott, where the radian measure is given as a flanking column in all the twelve tables dealing with the circular functions and their logarithms.

(7) A small table of Napierian logarithms.

Other tables of less importance are :

(8) *Antilogarithms.*
(9) *Square roots.*
} But pupils should be trained to use the logarithm table and the table of squares respectively for these purposes.

(10) For purposes of mensuration, a table of *cubes* is desirable. It can be used also for cube roots quite readily when the student has been shown how to look for the special cube root which he requires out of the three available for a given set of figures according to the position of the decimal point. Of course, it is the same kind of care as is needed in searching for a square root in the square or square root table.

Prof. Knott's tables contain also several other tables, one of the most useful giving, in one opening, from $n = 1$ to 100 the values of

$$\frac{1}{n},\ n^2,\ n^3,\ \sqrt{n},\ \sqrt{10n},\ \sqrt[3]{n},\ \sqrt[3]{10n},\ \sqrt[3]{100n}.$$

These tables, and those of Godfrey and Siddons, are, I believe, now the cheapest and for their price the fullest on the market, one costing 6d. and the other 8d., though, of course, Bottomley's tables still hold their own, but are bigger and dearer, and do not go so easily into the pocket.

Suggestions as to the First Introduction of these Tables to Students.

The tables which would first be used in junior classes would probably be logarithms, though, now that trigonometry is being taught so early, the tables of circular functions might in some instances come first.

With regard to logarithms, it is essential that pupils should never be allowed to forget that they are indices : that by their means all numbers can be expressed as powers of 10, and so can be multiplied or divided by mere addition or subtraction of the indices, and the result expressed again in the ordinary way either by the use of the antilog table or by the reverse use of the log table.

Also, the logarithms which have to be added should be written in *column*, never in rows : the rule being " Arithmetic in columns, Algebra in rows."

		logs.
Thus	$x = 37{\cdot}53 \times 52{\cdot}68$	1·5743
	$= 10^{3{\cdot}2960}$	1·7217
	$= 1977$	3·2960

so far as four figures can tell us.

The rule for the characteristic is that it is equal to the distance of the highest significant figure from the unit place, positive when it is to

the left of the unit place, negative if to the right. Pupils will soon learn to do this correctly, especially if they are occasionally reminded of the common sense of the rule. Thus $10^{3\cdot2900}$ lies between 10^3 and 10^4, that is between a thousand and ten thousand.

The tables, by the way, give the logs only of numbers whose highest figure is *in* the unit place.

It gives much greater reality to the work if the pupils have some practice in calculating for themselves.

Thus $10^{\frac{1}{2}}$, being $\sqrt{10}$, $=3\cdot16227766\ldots$,

and the pupil will find, on consulting the tables, that ·5000 is the logarithm of 3·162.

Also, $10^{\frac{1}{4}}$ = the square root of 3·162277...,

$10^{\frac{3}{4}}$ = „ „ 31·62277...,

and, similarly, $10^{\frac{1}{8}}$, $10^{\frac{3}{8}}$, $10^{\frac{5}{8}}$, $10^{\frac{7}{8}}$ can be formed, and verified by the tables.

These, with $10^0 = 1$,

$10^{-1} = \cdot1$,

$10^{-\frac{1}{2}} = \frac{1}{10}$ of $10^{\frac{1}{2}} = 10^{1\cdot5} = \cdot316\ldots$,

are sufficient basis for an excellent graph, which has the advantage of being a graph of logs or antilogs as one wishes.

Or, by taking a number of consecutive square roots, starting with $\sqrt{10}$, we obtain a series of numbers and their corresponding logarithms, and by judicious use of these by a series of contracted multiplications

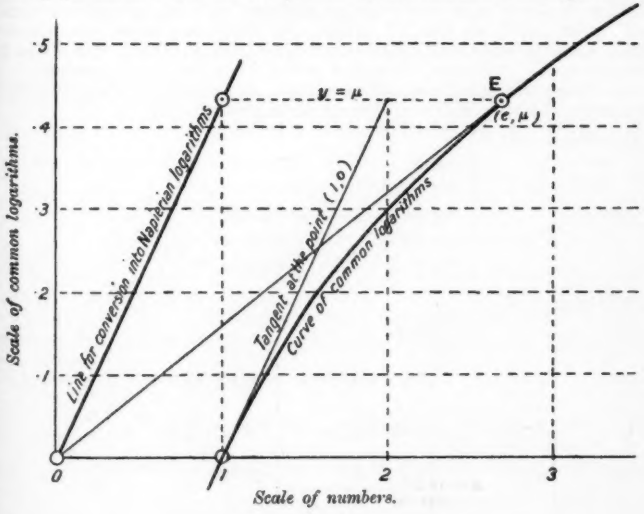

FIG. 1.

we can determine the logarithm of any number we please, or the number corresponding to any logarithm we please, this latter being very readily done by expressing the logarithm in a series of ascending powers of $\frac{1}{2}$.

If $\log 2$ is calculated, or accepted as ·30103, a great number of other logarithms can be at once written down, viz. those of 4, 8, 1·6, 3·2, 6·4, 1·28, 2·56, 5·12, and others that go into four or more figures; any logs so calculated being verified by the tables.

But the graph should not be neglected, as by its means it is easy to show how the decimal part of the logarithm is independent of the position of the decimal point in the number.

If the log curve is drawn from $x=1$ to $x=10$, and below that from ·1 to 1·0, reading the x scale 10 times magnified, and so on, we get a series of equal graphs one below the other, showing visibly that the decimal part of the logarithm depends only on the significant figures of the number.

A subsidiary scale consisting of a line through the origin at gradient μ would enable common logs to be changed into Napierian logs, and *vice versa*, the Napierian logs being the abscissae of the points on this line whose ordinates are the common logs; but that is a later development, and so also are the facts that the tangent to the curve at the point (e, μ) goes through the origin and that the gradient of the curve at (1, 0) is μ. But they are things which nothing but a graph can adequately visualise.

With regard to the other tables, it is needful to point out any peculiarities, such as that sines and cosines being less than 1, their logarithms are necessarily negative (except $\sin 90° = 1$, $\log \sin 90° = 0$).

And so also for tans of angles $< 45°$, or cotans of angles $> 45°$; whereas $\sec\theta$ and $\operatorname{cosec}\theta$ always exceed 1, and their logarithms are positive.

Also, pupils should be shown from the first that $\log \sin\theta$ is merely the log of $\sin\theta$, and could have been got by looking out $\sin\theta$ and getting its log from the log table. So many beginners are worried about natural sines and logarithmic sines, as if they were different sorts of sines. "Please, sir, which is it?" is a frequent question.

In solving triangles from ordinary data, some trouble is worth taking to scheme out an orderly condensed arrangement: *e.g.*

(1) Given a, B, C,

$a=$		$\log a$	$=$		
$A=$		$\log \sin A$	$=$		
$B=$		$\log 2R$	$=$	$\log 2R$	$=$
$C=$		$\log \sin B$	$=$	$\sin \log C =$	
		$\log b$	$=$	$\log c$	$=$

(2) Given a, b, c,

$s=$		$\log r =$					
$a=$	$s-a=$	$\log s-a =$		$\log \tan \frac{A}{2} =$		$\frac{A}{2} =$	
$b=$	$s-b=$	$\log s-b =$		$\log \tan \frac{B}{2} =$		$\frac{B}{2} =$	
$c=$	$s-c=$	$\log s-c =$		$\log \tan \frac{C}{2} =$		$\frac{C}{2} =$	
				$\log s$	$=$		
$2s=$	$s=$						$90°\ 0'$
	(check)	$\log s$	$=$				
				$\log r$	$=$		(check)
		$\log r^2$	$=$	(check)			

A Short Addendum on the Slide-rule.

Then comes the question of how widely the use of the slide-rule should be taught. I expect in most cases it is restricted to the Army and Science classes, in the upper school, and there pupils are allowed to use tables or slide-rules as they prefer.

In teaching their use, the fundamental unit of operation would seem to be the solution of the equation

$$x=\frac{ac}{b}, \quad i.e. \quad \frac{a}{b}=\frac{x}{c},$$

or

$$\log x-\log a=\log c-\log b.$$

The setting is as shown:

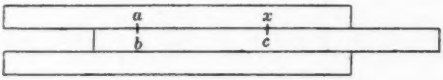

Fig. 2.

a and x being read on the top scale immediately over b and c, which are on the slide; and more continued operations are repetitions of this unit: thus

$$x=\frac{a/c_1/c_2/c_3/}{/b_1/b_2/b_3/},$$

all the work being done on the slide with the exception of the initial a and the final x.

This method gives the fewest movements of the rule,

viz. cursor to a on top scale;

then on the slide, b_1 to cursor, cursor to c_1;

b_2 to cursor, cursor to c_2;

b_3 to cursor, cursor to c_3;

.....

b_n to cursor, cursor to c_n.

Then read off x on top scale.

This requires one more factor in the numerator than in the denominator, and if this is not the case, it must be made so by one or more unit factors.

Similarly, if the sine scale is used,

$$x=\frac{a\,.\sin\gamma}{\sin\beta}$$

is the unit operation, sin 90° being used when a unit factor is needed.

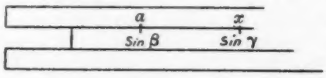

Fig. 3.

Notes on the Discussion.

Very little was said as to extent of tables desirable, but there were no dissentients to the scheme outlined above, and it was suggested that logarithms of numbers from 1·000 to 1·200 or thereabouts should be given in greater fullness and to more decimal places, partly because of their use in connection with compound interest, the greater fullness being desirable because of the large and varying proportional parts in this portion of the table.

A better way of dealing with compound interest would, however, be to have a compound interest table, such as is given in Dr. Knott's collection, from 1 to 50 years at a number of usual rates per cent.

Mr. C. S. Jackson insisted on the need for inculcating neat methods of work, neatness being almost more important than accuracy, in the sense that neat methods and clear figures properly placed are needful to ensure accuracy.

With regard to the examination difficulty, Mr. Dobbs suggested that pupils should be allowed to use their own books, which, however, is hardly practicable in a competitive examination. But possibly, for an extra fee, the examining bodies would under regulations supply a candidate with the tables to which he was accustomed, pending the time when the stock tables which they provide shall be such as to satisfy every reasonable requirement.

Obituaries.

Prof. F. R. Barrell, M.A.

The unexpected death of Prof. F. R. Barrell robs the University of Bristol of one of its most notable figures. He was a man of remarkable independence of character, and possessed of striking qualities of mind and heart that attached to him a host of friends. His gifts of ready sympathy and insight will long be remembered both by the colleagues to whom he was endeared and by the students who came under his inspiring influence. We quote from *Nature*, December 9, 1915, a passage that aptly sums the main characteristics of his teaching: "Sound, especially in fundamental conceptions, rather than brilliant as a mathematician, his main interests lay in the applications of mathematics to practical problems in physics and in methods of teaching, with constant insistence on the importance of a securer basis clearly and adequately grasped. His work bore fruit in its influence on those whom he trained."

We well remember the keenness of the pleasure that was given to him by his early work in the detection by X-ray photographs of the presence of foreign bodies in cases requiring surgical aid. Little we thought that the time would come when he would himself be called upon to apply in war what had proved so successful in its application to hospital work in peace. Throughout the summer vacation of 1915 his services were valued at Netley as those of a consummate master of the simple process he had invented, and just before his death the call reached him for his further services during the approaching Christmas holidays. But it was not to be. A more peremptory summons has taken from us a comrade whose worth and charm will not readily be forgotten.

Prof. H. W. Lloyd Tanner, F.R.S.

The following passage is taken from the Anniversary Address, to the Royal Society, delivered by Sir William Crookes, November 30, 1915:

"Mathematical Science has lost one of its most distinguished exponents by the death of Prof. Henry William Lloyd Tanner, to whose educational and administrative talents the University of South Wales and Monmouthshire is deeply indebted. His views on the mathematical training of students were thoroughly sound, and his services to education were by no means limited to his College or University. In addition to his teaching and administrative work at Cardiff he published many important investigations in mathematics, dealing firstly with the solution of partial differential equations, and secondly with the theory of numbers. These latter researches, which were distinguished by great ingenuity and originality, were not by any means completed when failing powers forced him to resign his professorship and cease work."

Over the well-known signature E. B. E. appears a notice in the *Proceedings* of the London Mathematical Society (Ser. 2, vol. 14, pp. xxxvii-xxxviii), from which we extract a few felicitous passages:

"He organised well, taught devotedly, and endeared himself to colleagues and to generation after generation of students. Given a difficult administrative problem, Tanner was the man to face it; given a wayward cutter of lectures, Tanner was the last to lose and the first to reclaim him; given a social effort in student life or a college entertainment to be got up, his was the youthful merry soul which boyish life made its ally. . . . He was a wise protector of others, but far too unsparing of himself."

He was respected for his intellectual honesty by all who knew him, and endeared to his many friends, no less by his geniality and ready wit, than by the inexhaustible sympathy of his unselfish nature.

THE WORK OF A LOCAL BRANCH.

By the late Prof. W. H. H. Hudson, M.A.

We meet to-day * to inaugurate a London Branch of the Mathematical Association. This is a step in a process of evolution.

The Mathematical Association is a continuation of the Association for the Improvement of Geometrical Teaching, commonly known as the A.I.G.T., founded in 1870.

I did not join the Association till 1883, but some of its earliest reports are in my possession; to these I shall refer.

It owed its existence to a profound wide-spread dissatisfaction with the lack of geometrical knowledge, and still more the absence of geometrical power, displayed by students who had passed through the ordinary school course of Euclid.

The first report, in January 1871, shows that Dr. Hirst was President; Archdeacon Wilson and Dr. Joshua Jones, Vice-Presidents; Dr. Wormell, Treasurer; Messrs. Levett and MacCarthy, Secretaries. There appears also a list of 12 Local Secretaries; this foreshadows what is going on now, the formation of local branches.

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* 27th November, 1909.

These were :

Messrs. Merrifield, Tucker,	London.	Messrs. Bushell,	Harrow.
Laverty,	Oxford.	Duckworth,	Tiverton.
J. F. Moulton,	Cambridge.	Eve,	Wellington.
Mann,	Edinburgh.	Miller,	Huddersfield.
T. S. Aldis,	Manchester.	Puckle,	Windermere.
		Richardson,	Winchester.

The first roll of members contained 43 other names.

The first officers of the Association may be accounted its founders, and as such are entitled to our reverence. Several have passed away. It would have been more fitting if some one of those who survive had filled the chair which I unworthily occupy to-day.

J. M. Wilson had written in the *Educational Times* in 1868, and about the same time Dr. Jones in a pamphlet, both powerfully in favour of reform. Messrs. Levett, Wormell and Tucker, subsequently secretary, had written letters to *Nature* in 1870.

The Association has had an uninterrupted succession of energetic and devoted secretaries, which continues to the present time.

At the same date, 1870, the British Association was also moving in the direction of the reform of teaching Geometry. As our business to-day concerns our own Association, I omit any further reference to the British Association, important as their proceedings were.

In the earliest writings the word Reform was generally employed. At the first meeting of the Association the word used in the title was changed to Improvement on the express ground that Reform might be ephemeral, but Improvement would always be needed. Can any one say that there is no need for it now?

From the first the main Improvement contended for was the discontinuance of the use of Euclid's *Elements* as the sole text-book of Geometry. This object has been attained ; Euclid has been dethroned, but it may still remain a question whether all the Improvement that was hoped for has or has not taken place. It was admitted in the arguments that were used that the main fault was that of learning by heart the exact words of a particular translation of Euclid's book. If this fault has not been corrected, the mere substitution of another text-book can avail but little. The worship of the letter killeth.

Messrs. Levett and MacCarthy say, in December 1873, that mathematical teachers "knew from their own experience that the majority of those who 'learnt Euclid' never exercised in the study any faculty except the memory, and that they not only failed to solve for themselves geometrical exercises, but were unable to reproduce a train of reasoning that was set before them unless they were supplied with the very words in which to exhibit the argument."

And in another place : "The average boy is capable of nothing further than the reproduction of a number of propositions in the words of his text-book ; he breaks down when he is asked to do the simplest deduction" ; and they go on to attribute this to the retention of Euclid as a text-book.

Was it not, however, due to the abuse of the text-book ? It will be interesting to know whether the present generation of teachers find that better results follow from the use of the modern text-books that have been adopted in place of Euclid's. If they are learnt by heart as Euclid's was, I shall be surprised if there is any improvement.

The earliest effort of the Association was to produce first a syllabus of a geometrical text-book, issued privately in 1873, and in 1875 to the public, followed by proofs of the propositions in the Syllabus, the

first part of which appeared in 1883 and the second in 1886. Simultaneously with the preparation of these proofs, which form an Elementary Treatise, they were busy endeavouring to induce examining bodies to admit other proofs than those of Euclid, and to test knowledge as well as memory. In 1877 they sent a letter of application to a large number of examining bodies asking that their examinations should be guided by the Syllabus of the Association.

In these endeavours they had some success, but they did not get all they asked for. Oxford was for a long time immovable, but even Oxford gave way at last. I remember Mr. Langley showing me with pride the first Rider that he believed had ever appeared in an Oxford Smalls Geometry paper. I am not certain of the date, but it must have been after 1892.

The Association gradually enlarged its domain. It added to its work in 1878 the subjects of Higher Plane and of Solid Geometry and of Geometrical Conics, very natural and proper extensions, entirely in accord with its title. Subsequently, in 1883, it took up the subject of Arithmetic, one of its prominent members explaining that this was quite justifiable, since Arithmetic was necessary to the measurement of the earth.

In 1886 Mechanics began to occupy its attention. It has now absorbed the whole field of School Mathematics. This being the case, and the first object, the deposition of Euclid, having been practically accomplished, the Association in 1897 shortened its name to that which we now adopt—The Mathematical Association.

This brief historical retrospect shows that the aim of the London and other branches of the Mathematical Association is to continue the work of endeavouring to improve Teaching, and indicates that its scope does not include endeavours to extend the boundaries of mathematical knowledge. This function is appropriate to the London Mathematical Society.

The extension of the activities of the Association from the improvement or reform of teaching Geometry to that of Mathematics in general does not preclude the consideration of the teaching of what used to be known officially as Natural Philosophy and colloquially as Mixed or Applied Mathematics, but now is more commonly included under the title of Physics. The subject of Mechanics has already been annexed.

It would be a retrograde movement to limit the Association to Pure Mathematics. The dissociation of Pure and Applied Mathematics, which is a distinguishing feature of the regulations of the University of London, is calculated to have a prejudicial effect on the elementary study of both.

It has been one of the difficulties of mathematical teaching that the school-boy's mind, reflecting perhaps the opinions of ignorant elders who ought to have known better, is often imbued with the belief that Mathematics are useless. Elementary Mechanics is a great aid in exorcising that spirit of unbelief. I think I have noticed that practical Mechanics in the use of the bicycle has been of service in this respect.

Greatly increased attention has been given of late years to the *Theory* of Teaching. It may well come within the scope of this Association to consider the doctrines of the experts on this subject, so far as they are applicable to the teaching of Mathematics, including Physics; and, when their exposition has been clouded by the use of metaphysical and psychological, technical but unfamiliar, terminology, to translate them into a language understanded of the people who have to exercise the Art of Teaching.

The use of the fingers in education has been impressed on us by the success of the Kindergarten in the training of very young children, and by the improvement in reasoning capacity that has been shown by those who in actual workshops have learnt the necessity of accuracy, and have been accustomed to deal with things instead of or as well as remembering words. The way in which this may be utilised in the teaching of Mathematics and its limitation are properly included in the purview of our Association.

I say its limitation, for there is a real danger of its being carried too far. It is too interesting. Greatly as interest facilitates the advancement of knowledge, progress can seldom be made if *only* what is interesting is to be studied. It is in the incipient and the advanced stages that interest is of most importance. In the incipient there is little else to appeal to, in the advanced there is too much else to distract. In the middle or school-boy stage, in order to prepare for the duties of life, one of the lessons that should be learnt is the subjection of inclination to duty.

Considering specially the subject of Geometry, all agree with the propriety of beginning with the use of ruler and compasses, which was neglected by those who learnt Euclid's propositions by heart and didn't know that Euclid wrote about Geometry. It is a question that needs solution in the practical working of every school to decide how far and for how long this preliminary practice should continue, and to what extent it should be carried on simultaneously with the study of geometrical relations in their theoretical aspect as propounded by Euclid or any other writer.

Another question concerning this subject can scarcely be considered to be settled, and may perhaps still form a suitable subject for discussion by the Association. Are we to depart from "the purity of the ancient geometry by transforming geometrical theorems into algebraic formulae, that is to say by substituting in place of concrete magnitude—such as lines, angles, superficies, volumes—their respective measures"? I quote from a document issued by the Italian Government in 1871, quoted by Dr. Hirst in his Presidential Address in 1872. Dr. Hirst referred to this as "striving after a deceptive facility of treatment by the introduction of algebraical processes in place of geometrical reasoning."

Is there not something to be said on the other side?

It is interesting to note that Dr. Hirst, in this address and that of the previous year, referred to the dissatisfaction in France and Italy with the text-books that had been used in place of Euclid's, and mentioned that the Italian Government in 1867 felt themselves obliged to re-establish the use of Euclid's *Elements* in the Gimnasi and Licei in order to "purge from Italian schools the many worthless books which private enterprise had succeeded in introducing." The use of improved text-books is a result which we in England may hope for from the mutual assistance which mathematical teachers may give one another at meetings such as those of our Association rather than from Government interference.

A propos of text-books, I may mention that this Association has a small library, which the formation of a London branch may have the effect of bringing into greater usefulness than in the past.

A question I have been often asked and scarcely ever been able to answer satisfactorily is, at what age this or that subject or part of a subject should be studied? An allied question is, what is the order in which mathematical subjects should be taken up? and what is the best manner in which they should be interwoven? for I take it that it is now generally accepted that the watertight compartment

doctrine is or deserves to be broken down. The use of letters in Arithmetic, of geometrical diagrams in Algebra, of the Differential Calculus in Algebraic Geometry, formerly tabooed, are now not only permitted but encouraged. Many questions of detail arise out of this extension.

The list of questions of detail that from time to time might receive the consideration of the Association could be indefinitely extended: improvements in processes, in arguments, in nomenclature, in definitions, and so on. Matters of detail, though they may seem of minor importance, affect very powerfully the success or failure of any art.

Since it may be expected that the London Branch will meet oftener than once a year, which has been the practice of the Association, I feel justified in suggesting matters of detail as worthy of being included in the scope of the work of the branch. Particular problems of interest need not be excluded.

Nor can we neglect the subject of examinations, though examinations should be kept out of sight in teaching.

Experience shows that, even if they are disregarded by the teacher, the eye of the pupil will be only too fixedly directed to them.

Where the examination schedule exercises a pernicious influence on study, such as those in Geometry, which our energetic predecessors have more or less succeeded after long struggle in ameliorating, this Association may with propriety by protest and entreaty continue that good work.

Where the schedule is satisfactory, and sometimes it is the more satisfactory the less it is detailed, the examination papers may be at fault, they may be of too great length, a very common mistake in Cambridge, or they may tend themselves too readily to the encouragement of cram, or they may have some other heinous fault.

In some such cases it might be fitting for the Association to call attention to it.

The examinations of the University of London naturally fall under the notice of the London Branch, and it is possible that the regulations and advice to teachers of the London County Council and the Board of Education might merit attention. Representations to public bodies would, however, come with more propriety from the whole Association than from a single branch.

There is another aspect of our meetings. We are, or have been, all engaged in the teaching of Mathematics, and we wish to improve that teaching, our own as well as other people's. We meet for mutual improvement. United by this bond, there will arise as we meet together a feeling of comradeship in the battle in which we are engaged, fighting always against ignorance, often against idleness, and sometimes against prejudice.

This camaraderie may be further promoted if a social character, at all events in part, is given to our meetings. For some years our annual meeting has been accompanied by a dinner. A distinguished friend of mine, well known to most of you, has told me of an ancient mathematical society to which was attached the soubriquet of "The 3 P's," because each man had his Pipe, his Pot and his Problem.* We shall not precisely imitate that, but I see that Tea is included in the agenda.

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*The Mathematical Society, 1717-1845. "Its habitat was Spitalfields, and I think most of its existence was passed in Crispin Street. It was originally a plain society, belonging to the studious artisan. The members met for discussion once a week; and I believe I am correct in saying that each man had his pipe, his pot and his problem." De Morgan, *Budget of Paradoxes*, Vol. I. p. 376. 2nd Edition, 1915. [W. J. G.]

SUGGESTIONS FOR NOTATION AND PRINTING.

By Prof. G. H. Bryan, F.R.S.

The list of suggestions issued by the London Mathematical Society is open to several amendments and possibly improvements.

1. The use of small fractions in the midst of letterpress is often open to the objection that such fractions are difficult to read, and, moreover, very often do not come out clearly in printing. It is especially difficult to distinguish $\frac{1}{3}$ from $\frac{1}{8}$, and the latter sometimes prints exactly like the former if the type does not ink properly or is slightly uneven. For this reason it would be better to confine the use of these fractions to such common forms as $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{3}$, and to use the notation 18/22 for other fractions; at any rate small fractions should not be used for *both* thirds and eighths, and as thirds are usually commoner, it would be well to restrict their use to these.

2. Most mathematicians already use i or ι for the square root of -1, so that the recommendation is hardly necessary. On the other hand, there are many cases in elementary text-books where these symbols cannot be used. The objection to $\sqrt{-1}$ is the bar over the -1. All that is necessary is to omit this and write the symbol $\surd -1$. I always do this (but the printers have a habit of inserting the bar).

3. While referring to i, may I point out the necessity of a symbol to represent $\log -1$ in calculating the sums of products of quantities, some positive and some negative, by the use of logarithmic tables? The need of a very simple notation has arisen in connection with the evaluation of the Fourier coefficients of aeroplanes, and we are now using the prefix j for the purpose. It stands for $\log -1$ plus the number which follows, thus $\log -2 = j\,0{\cdot}30103$ and $\log -0{\cdot}02 = j\,2{\cdot}30103$. Where an even number of j's occurs in the logarithms to be added together (or subtracted), these cancel, and the product is positive. Where the number of j's is odd, there is one j left in the logarithm, and the product is negative. For our aeroplane calculations this notation was absolutely indispensable.

4. In most cases the notation $n\,.\,n+1\,.\,n+2$ for $n(n+1)(n+2)$ is quite free from ambiguity. All depends, however, on the class of calculations concerned.

5. There is no need for a bracket in $n!^2$, since the factorial of n^2 would be distinguished by $n^2!$. Again, $n+1!$ and $2n!$ are usually sufficiently distinctive, especially if a stop is used in $2^n\,.\,n!$, as we should then use $2\,.\,n!$ to represent twice factorial n.

6. To avoid the use of small fractions it may be better to print

$$\frac{p+q/2}{r/3+s/4} \quad \text{instead of} \quad \frac{p+\frac{1}{2}q}{\frac{1}{3}r+\frac{1}{4}s}.$$

7. The recommendation ignores the useful symbol exp for exponentials or powers of e. This can often be used with advantage, notably in the examples shown:

$$\exp -\frac{n\pi x}{c} \quad \text{and} \quad \exp -\frac{x^2}{4kt}$$

are often much better than $e^{-n\pi x/c}$ and $e^{-x^2/4kt}$.

The exponential notation can also be used for powers of other bases where the index is complicated ; for example, the notation

$$\exp_a \frac{x^2}{4c^2t} \quad \text{for} \quad a^{x^2/4c^2t}$$

has been legitimised by previous use.

8. It is quite useless to recommend the substitution of dashes for dots in the fluxional notation for velocities and accelerations, because dashes are so often used for other purposes. For example, θ'' might mean an angle of θ seconds. The only rational plan of avoiding the printer's difficulties is to place the superior dots *after* the letters instead of over them ; thus $x'y'' - y'x''$ is all that is necessary.

9. The circular issued by the London Mathematical Society contains a specimen of printing showing all the leads and spaces used in setting up a piece of work involving mathematical formulae. As a matter of fact, however, this specimen is highly misleading, because the letterpress is "leaded," the lines of printing being spaced out by strips of lead inserted between them. Consequently, very few of the black lines and marks represent spacings introduced by the use of mathematical formulae. To make things clear, the example given should either have been printed in "solid" type (*i.e.* without spacing out the lines of letterpress), or the additional leads introduced for this purpose should have been left blank instead of being printed black. It is quite impossible to form a good idea of the difficulties encountered by the printer in setting up formulae from such an example. G. H. BRYAN.

MATHEMATICAL NOTE.

462. [L¹. **14. a.**] The square of the major axis of a conic inscribed in a triangle is equal to the sum of the squares of the radii of the director circles of the inscribed conics whose centres are the foci of the original conic.

Take the triangle as triangle of reference, and let (x, y, z), (x', y', z') be the areal coordinates of the foci. Let λ and μ be the semi-axes of the conic, and ρ, ρ' the radii of the director circles.

Then $\mu^2 = \alpha\alpha' = \beta\beta' = \gamma\gamma'$ (trilinear coordinates).

The square of the distance between the foci

$$= 4(\lambda^2 - \mu^2) = \Sigma bc \cos A\,(x - x')^2.$$

The middle term of the second expression reduces to $-4\mu^2$.

$$\therefore\ 4\lambda^2 = \Sigma bc \cos A\,(x^2 + x'^2).$$

Now, if H is the orthocentre and P the point (x, y, z),

$$PH^2 = \Sigma bc \cos A\,(\cot B \cot C - x)^2$$
$$= -4R^2 \cos A \cos B \cos C + \Sigma bc \cos A \,.\, x^2,$$

so that $\Sigma bc \cos A \,.\, x^2$ is the square of the radius (ρ) of the director circle of the inscribed conic, centre P, for this circle and the self polar circle of the triangle cut orthogonally.

Hence finally, $4\lambda^2 = \rho^2 + \rho'^2$. Q.E.D. N. M. GIBBINS.

REVIEWS.

Chappell's Five-figure Mathematical Tables. By E. CHAPPELL. Pp. xvi + 320. 5s. net. 1915. (Chambers.)

Mr. Chappell's book of five-figure tables contains several new and useful features. In addition to the usual tables of logarithms and antilogarithms (called by the author "illogs") and of trigonometrical functions and their logarithms, tables are given of cologarithms, the logarithms of logarithms (called lologs), and the antilogarithms of the antilogarithms of numbers (called illologs). These tables should prove extremely useful in all calculations involving involution and evolution. The lolog table gives the lologs of numbers from 0·00100 to 1000. In the case of the numbers below unity, the lologs are the logarithms of the numerical values of the logarithms of the numbers, and the results are printed in red. Thus, since $\log 4 = 0{\cdot}60206$ and $\log 0{\cdot}25 = -0{\cdot}60206$, the lologs of 4 and 0·25 are each $\bar{1}{\cdot}77964$, but the former is printed in black and the latter in red. This gives rise to the rule, "the lologs of a number and its reciprocal are numerically equal but opposite in colour." The same distinction by colour is made use of in the illolog table, and illologs are given of all numbers, black and red, from $\bar{6}{\cdot}0$ to 0·5. The great usefulness of these tables, and more especially the "red" tables, can be seen by comparing the labour involved in the following example taken from the introduction with that which is necessary when only an ordinary logarithm table is available.

To evaluate $(0{\cdot}4123)^{1{\cdot}406}$:

$$\begin{aligned} \text{lolog } 0{\cdot}4123 &= \bar{1}{\cdot}58522 \\ \log 1{\cdot}406 &= 0{\cdot}14799 \\ & \overline{\bar{1}{\cdot}73321} \end{aligned}$$

The illolog of $\bar{1}{\cdot}73321 = 0{\cdot}2877$, which is the value required. In this case the lolog is printed in red, and consequently the illolog is taken from the "red" table.

In the tables of Trigonometrical functions the author has adopted the excellent plan of giving the functions and their logarithms in consecutive columns. The pages are headed with the number of degrees, the minutes are given in the first column, and after the value of any function the proportional part for one second is given, thus reducing the labour of interpolation to a minimum. Each line contains the sine, log sin, tan, log tan, sec and log sec of the corresponding angle, and consequently also the corresponding "co"-functions of the complementary angle.

The printing is good, all the tables are well and clearly arranged, and the introduction is a model of clear exposition. The book should prove of great value to all whose work demands the use of tables.

Table of Compound Interest at ¼ per cent. and of Antilogarithms to 60 figures to base 1·00125. By J. J. STUCKEY. Pp. 117. £1. 1s. net. 1915. (Allen & Unwin.)

The main table consists of the values of $(1{\cdot}00125)^{\pm n}$ to 60 places of decimals for all integral values of n from 1 to 2000, and is thus in effect a table of antilogarithms of positive and negative integers to the base 1·00125. Such a table constructed to a base differing little from unity is undoubtedly useful in certain types of calculations, but it is difficult to see any special advantage in this particular base rather than, say, 1·001. Since interpolation has to be employed for all values of n which are not integral, the labour in using the table beyond six figures is prohibitive. Several auxiliary tables are given by means of which logarithms to the base 10 or e may be calculated. In the introduction the author attempts to give an exposition of the method of construction and application of these tables, but until this is rewritten the book will be practically valueless. Explanations are either entirely lacking or so obscure that only those with leisure and great patience will ever succeed in following the author through the mazes of figures in the examples. Of the eighteen illustrative examples, two show how such functions as $\tan^{-1}x$ and $\sin x$ can be calculated, but these are not fair tests of the use of the tables, as the values of x are so chosen that n is integral; others are intended to

show how certain logarithms can be calculated, but these we have frankly been quite unable to follow; the remainder are simple calculations, such as $(1{\cdot}035)^{25}$ and $2\frac{1}{2}$, and direct and inverse questions on interest—common enough to those engaged in actuarial practice, but quite unintelligible to the ordinary mathematical reader without an explanation of the symbols employed. Thus one of the inverse examples is to find i from the equation

$$120 = 2{\cdot}25a_{50} + 112{\cdot}5v^{50}.$$

The solution is distinctly neat, but would appeal to a wider circle of readers if it were explained that $a_{50} = \frac{1-v^{50}}{i} = \frac{1-(1+i)^{-50}}{i}$, where i is the rate of interest per unit per period. The examples of direct calculation are far from impressive. Thus, to obtain $(1{\cdot}035)^{25}$ we find by inspection of the tables that the logarithm of 1·035 lies between 27 and 28, and, using interpolation (necessitating the use of a slide rule or an ordinary logarithm table), that 25 times this logarithm is 688·45 approx. The next process is to take the antilogarithms of 688 and 689 from the tables and interpolate again, and the result is finally obtained correct to 5 figures. The work can be performed in one-tenth of the time by using Chappell's tables. In fact, all the examples given, except those on logarithms to 12 or more places, can be solved very much more rapidly and with sufficient accuracy for all practical purposes by Chappell's tables. S. T. SHOVELTON.

A Course in Fourier's Analysis and Periodogram Analysis for the Mathematical Laboratory. By G. A. CARSE and G. SHEARER. Edinburgh Mathematical Tracts, No. 4. Pp. viii + 66. 3s. 6d. net. 1915. (G. Bell and Sons.)

The first chapter of this tract contains an elementary account of Fourier's series treated from the analytical point of view, while in the second, methods are explained by which the coefficients in a series can be practically evaluated when the relationship between the dependent and independent variables is given in the form of a curve or table. The greater part of the matter in these chapters has already appeared in the Napier Tercentenary volume on "Modern Instruments and Methods of Calculation," but the discussion in the tract is confined to arithmetical and graphical methods. In Fourier Analysis it is assumed that the period of the total fluctuation of the function to be analysed is known, but when this is not the case, we are confronted with a much more difficult problem. There are many natural phenomena, such as the spottedness of the sun, the brightness of variable stars, the magnetic elements of the earth, whose magnitudes exhibit a quasi-periodicity, and it is a matter of great interest to attempt to discover whether these variations can be represented as the sum of a number of simple periodic terms. The case of the tides shows us that, supposing such a mode of representation to be possible, the arguments of the various terms involved may not stand in any simple relation to one another. Lagrange seems to have been the first to consider the problem, and he invented two methods of dealing with it, of which the first appeared in 1772 and the second in 1778. The next step was taken by Schuster, who in 1897 proposed the use of the Periodogram for the purpose of detecting periodicities. This method and a worked example illustrating its application is explained in Chapter III. of the tract. It is certainly an oversight that Schuster's name is not mentioned. In 1914 Dale proposed a method which is fundamentally the same as the second Lagrangian one, and this is also fully explained, and the results obtained by means of it are compared with those obtained by the Periodogram. The agreement is very satisfactory. The amount of computation involved in the application of either method is considerable. It is probable that the Periodogram will prove to be the better method for detecting real periodicities when they are combined with purely fortuitous fluctuations, but in other cases the Lagrange-Dale method will effect the analysis more rapidly. The final chapter deals with Spherical Harmonic Analysis, and contains an account of Bauschinger's development of Neumann's method for calculating the coefficients. It is obvious that the tract is one of great value to students of astronomy and cosmical physics. J. B. DALE.

Elementary Algebra. First Year's Course. By FLORIAN CAJORI and LETITIA R. ODELL. Pp. viii + 206. 3s. net. 1915. (Macmillan Co., New York.)

At the present time elementary algebra is a source of anxiety to many teachers. What is the object of teaching algebra? Do we teach algebra for its own sake, or as a necessary preparation for more advanced mathematics, or as collaterally useful to students of physics and engineering? Ought we to adopt a tolerably strict standard of logical reasoning? Ought we to rely largely on concrete (some would even say utilitarian) examples to teach the arts of algebraical manipulation? Ought we to impose abundant drill in abstract algebraical operations, hoping that the meaning and use of our proceedings will become clear as we go along? "C'est en forgeant qu'on devient forgeron." Is there not something to be said for the dictum that the answer to each of the latter questions is in the affirmative, provided the teacher has some sense of proportion?

This, at any rate, is apparently the view taken by the authors of this attractive little book.

Professor Cajori is a well-known authority on the history of mathematics, and his learned and scholarly bibliography of the slide rule has settled several doubtful claims. Hence it is not surprising that one of the novel features of the present work is the stress laid on the history of the subject.

Excellent portraits are given of Newton, Euler, Vieta, and Descartes. A facsimile of the passage in the *Whetstone of Witte*, proposing the sign = for equality, is of special interest. It is true that the historical notes are in themselves but scanty. However, we know but little of the road to the mind of a child, and those who look back on their own childhood will acknowledge that some of the ideas recalled most vividly in after life are not those which were most earnestly impressed upon us in youth. At any rate, a look at these portraits may lodge an idea capable of growth in some child's mind. Moreover, these notes justify a teacher in developing them somewhat, without incurring the imputation of teaching "something that isn't in the book."

That specially hateful type of example—the pseudo-utilitarian, or sham-practical—is completely absent, and altogether this is a sane and well-balanced volume. C. S. J.

An Introduction to the Theory of Automorphic Functions. By LESTER R. FORD. Edinburgh Mathematical Tracts, No. 6. Pp. viii + 96. 1915. (Bell & Sons.)

The subject of automorphic functions is an offshoot of the theory of linear differential equations of the second order. If x is a complex variable and y_1, y_2 are the two solutions of the equation $\frac{d^2y}{dx^2}+Qy=0$, where Q is a function of x, then when x describes any closed path, y_1, y_2 are transformed into $a_1y_1+a_2y_2$ and $b_1y_1+b_2y_2$, where a_1, a_2, b_1, b_2 are constants such that $a_1b_2-a_2b_1=1$. Thus, if $y_1/y_2=s$, s is transformed into $(a_1s+a_2)/(b_1s+b_2)$: the transformations corresponding to the different possible closed paths form a group, the group of the differential equation.

Now let s be taken as independent variable and x as a function of s. Just as a periodic function, say $\tan\theta$, is unaltered by the addition of a period, π, to the variable θ, so x is unaltered when s undergoes one of the linear fractional transformations of which the group consists, and x is called an automorphic function of s: if it is to be a one-valued function, the differential equation must satisfy certain conditions.

The theory becomes more attractive when with Poincaré we start from a given group instead of a given differential equation. The problem of finding a suitable group is something like that of making a skeleton design for a non-Euclidean wall-paper or carpet, and when the group is given the corresponding automorphic functions can be expressed by means of infinite series, somewhat after the fashion of the partial fraction series for trigonometric and elliptic functions, which are indeed very special examples of automorphic functions.

Elliptic functions afford two other illustrations. A function of u, with periods 2ω, $2\omega'$, if one-valued, is also one-valued when regarded as a function

of $z=e^{\pi u/\omega}$, and is moreover unaltered if z is multiplied by any power of the constant $e^{2i\pi\omega'/\omega}$: it is then an automorphic function of z, whose group consists of the powers of a single operation. Better examples are afforded by the modular functions; for instance, if in Jacobi's notation k^2 is treated as a function of K'/K, it is unaltered by a group in which all the coefficients a_1, a_2, b_1, b_2 are integers.

The example of the modular functions brings up the theory of hypergeometric functions in general, and their relation to the spherical triangle, also the theory of the five regular solids, and some very elegant parts of the Theory of Numbers, such as the Pellian equation and theory of quadratic forms.

Again, automorphic functions, having been introduced for the sake of linear differential equations of the second order, are found to be useful in a much wider sphere. By their use any two variables connected by a single algebraic equation may be treated as one-valued functions of a single variable, and further, the functions that satisfy any linear differential equation whose coefficients are algebraic functions of the independent variable may be so expressed at the same time as the independent variable and the coefficients.

It will be easily understood that any one writing a tract of ninety pages on a subject with such wide ramifications must choose between undue compression and judicious omission. Mr. Ford has chosen the latter, and has devoted most of his space to a careful treatment of introductory matters, thereby probably making his book more useful; for the student who wishes to go further, he has provided a very full bibliography. A. C. DIXON.

A Twentieth Century Arithmetic. By C. S. JACKSON, F. J. W. WHIPPLE and LUCY ROBERTS. 4s. 6d. 1915. (J. M. Dent & Sons.)

This book, a thoroughly sound and sincere piece of work, differs in several important respects from various recent Arithmetics.

The most striking point, perhaps, is the arrangement of work advocated and illustrated in the worked examples.

Some teachers advocate (for problems in Arithmetic which are more than mere computation) a reasoned statement occupying, say, two-thirds of the page completed by the necessary multiplications and divisions worked in the remaining third, usually the right-hand side. This arrangement has the advantage of being likely to produce a clear statement, but the defect that the work of computation tends (if the teacher is not careful) to be written too small, too untidily, or at worst on odd scraps of paper. Other teachers (and these are they to whom this book is likely to appeal) place statement and computation together in the centre of the page. This method makes it very much more difficult to keep the statement clear (the specimen examples prove this possible), but it tends to produce computations writ large.

Again, teachers differ in the relative stress laid on mensuration and money sums. Those who regard mensuration as nearly half of Arithmetic, and like to teach early such things as the volume of a cylinder (round tins are so common) and the determination by Pythagoras' theorem of the height to which a ladder will reach, will regret that these are postponed to Bills of Exchange and Stocks. One may say that the book is intended for the clerk rather than for the engineer.

The vexed question of fractions or decimals first is here settled in favour of fractions, and the authors throughout show considerable affection for the "method known as practice," an affection which the reviewer shares.

In multiplication and division of decimals "standard form" is not used (though a mention of it is conceded in connection with division).

This is a vexed question indeed! It is perhaps safe to say that those who regard "standard form" as the best method will probably consider that here used as the second best.

To come more to details. The book is in three parts. The first deals with the four rules, reduction and factors; and finishes with an interesting chapter, "Easy problems: introduction to Algebra." Rectangular areas and volumes are commenced in this part.

Part II. commences with a chapter on fractions, and then come decimals (introduced by decimal coinage and measures).

The explanations of what is meant by the multiplication of two fractions or two decimals are singularly clear and happy.

Then come compound practice (including rectangular areas and volumes) and unitary method, and Part II. ends with a long chapter on the applications of proportion.

This, under its various titles, is usually the most interesting part of an Arithmetic, and it is with a sigh of relief that one sees that the clumsy unitary method (beloved of beginners) is replaced in the specimen examples by the use of ratios.

Of the more hackneyed part of this chapter one may say that the treatment given assumes the subjects treated to be easier than boys usually find them, and will need to be supplemented ; but the compression applied to this part has left room for some very interesting and useful work in connection with insurance and taxation.

Part III. contains first the harder part of commercial Arithmetic, compound interest, discount, stocks ; then a chapter on approximate calculations and percentage error ; and then a set of chapters which contain much very easy work, which may with advantage be read with Part II., on angles, time, lengths, mensuration, squares and square roots, and graphs.

The graphs drawn as specimens are particularly attractive, since, unlike so many graphs in print, they obey the laws that the scales should be clearly marked along the axis and that no graph should lack a title : in short, they really explain themselves.

A chapter on logarithms and a set of revision examples (about 300 in all) conclude the volume.

The examples throughout are printed in short sets (usually seven examples each), and there are some interesting sets included containing suggestions for appropriate practical work.

The explanations throughout the book are clear, and do not disdain humorous touches, *e.g.* "The holder of government bonds sleeps well and dines badly—while the holder of mining shares dines well and sleeps badly."

It is interesting to notice that the German system of coinage is not mentioned, though there is a good deal of information as to the "coinage of the principal civilised states."

C. O. TUCKEY.

Statics. By R. C. FAWDRY, M.A. Pp. 305. Part I., 2s. 6d. ; Part II., 2s. ; Parts I. and II. in one volume, 4s. 1915. (G. Bell & Sons.)

This book contains a large number of excellent examples, of which many are of a practical type. In writing the book, the author has had in view the needs of "Woolwich candidates, engineering students and those reading for scholarships in mathematical and mechanical sciences." The course provided satisfies those needs. The calculus is not used. The scope of the work may be indicated by saying that, starting with the elements of the subject, it contains in its later parts sections on bending moment and shearing force, frames, link-polygons and the conditions of equilibrium of forces in three dimensions.

In accordance with what is now becoming a tradition in elementary text-books, the exposition of the main theorems in the earlier parts is left incomplete. Thus in the first chapter, which deals with moments, the description of the second experiment—two unequal weights balancing on a metre-scale supported at its centre of gravity—*ends* with the direction : "In each case multiply the weight by the distance of its centre from the fulcrum and compare the results." The result of this comparison is not formally stated, except by inference on page 33, where further deductions are made from it. The book also contains many instances of that modern use of the note of interrogation which may irritate one seeker after truth and depress another, but which does very little towards enlightening either of them. These features of the book are not so noticeable in the later parts.

There are some errors for correction in future editions, *e.g.* μ suddenly introduced on page 260, a rotation spoken of as equivalent to a couple on page 289, and, on page 261, an extraordinary statement of a problem which is worth giving here in full : "A cylinder of radius r is held between two rough boards inclined to one another at an angle 2α. Prove that the cylinder cannot

be squeezed out unless it is at a distance from O less than $r \operatorname{cosec} \alpha$." Many corrections have to be made in this before any sense can be made of it. A full solution of the problem thus concealed by language is given in the text.

Lessons in Geometry. By CHARLES M'LEOD. Pp. 507. 1915. (Aberdeen, The University Press.)

Elementary Geometry. By W. E. PATERSON and E. O. TAYLOR. Pp. 327. Part I., 1s. 8d.; Part II., 1s. 8d. 1915. (Clarendon Press.)

In the eighty lessons and six appendices contained in his book, Dr. M'Leod has given us an exceedingly interesting and stimulating account of his subject. The work has a wide scope; the course of geometry is complete from the very beginning, and by the time the last appendix has been mastered, the reader has been through the whole of the ordinary elementary work in plane and solid geometry, in plane trigonometry, in logarithms and in mensuration, and has also learnt something about conic sections and derivative functions. A large number of examples is provided. A paragraph from the preface explains the use to which the book may be put.

"With junior pupils the book requires the guiding hand of the teacher. Young teachers may find in it suggestive hints from an old hand. More advanced pupils may derive profit from reading through its pages on their own account."

This estimate is more than justified by an examination of the book. Any teacher of elementary mathematics would find it useful to him, and one can hardly imagine a better text-book for a student who is beginning the study of geometry somewhat late in his educational career.

The book by Messrs. Paterson and Taylor is a school text-book on the usual lines. It presupposes a preliminary course, and is specially adapted to follow Mr. Taylor's *Introduction to Geometry*, also published by the Clarendon Press. The proofs of the first eleven theorems are placed in an appendix and are not to be learnt till some considerable number of the succeeding theorems have been mastered. A special feature of the work is that it is divided into Chapters, not into Books. The Theorems (of which there are 77) are thus numbered consecutively, and the same method is used with the Problems. The proofs are sometimes given in outline only, and the book is quite in the fashion in the number of question marks and inconclusive paragraphs which it contains. The printing is good, and the volume would be found quite workable as a text-book for classes which are being taken through the ordinary elementary course in Plane Geometry.

A First Book of Arithmetic. By S. LISTER. Pp. 258. 1s. 6d. 1915. (Macmillan & Co.)

This handy little book is one of the series called "First Books of Science." It will meet the requirements of any teacher who is looking for a guide for his pupils through a not over-elaborated course of Arithmetic. A special feature of the earlier chapters is the large amount of ground covered without using either long multiplication or long division. The book is essentially a "First Book," and does not deal with compound interest, stocks and shares, contracted methods used in approximation, and the like. Letters are used for numbers wherever such a course is advisable. The large clear type used for numbers is a good feature of the book.

A Review of High School Mathematics. By W. D. REEVE and R. SCHORLING. Pp. x+70. Price 40 c. 1915. (University of Chicago Press. Cambridge University Press.)

The authors of this book are instructors in Mathematics in the University High School of the University of Chicago. The "review" consists of a number of questions covering the courses for the first and second years in Algebra and Geometry, and for the third year in Algebra. These questions are all of a straightforward kind, and enable one to get an idea of the programme of work in the school in which the authors teach. A tentative scheme for a shorter course is outlined at the end of the book; this course and the review questions in the body of the work will be of interest to any English sympathiser in the authors' "effort to initiate a plan for the standardisation of high-school mathematics."

T. M. A. COOPER.

THE MATHEMATICAL ASSOCIATION OF AMERICA.

On December 30 and 31, 1915, there was held at Columbus, Ohio, the organisation meeting of a new national mathematical association, the call for which had been signed by 450 persons representing every state in the Union, the District of Columbia, and Canada. The object of the new Association is to assist in promoting the interests of mathematics in America, especially in the collegiate field. It is not intended to be a rival of any existing organisation, but rather to supplement the Secondary Associations on the one hand, and the American Mathematical Society on the other; the former being well organised and effective in their field, and the latter having definitely limited itself to the field of scientific research. In the field of collegiate mathematics, however, there has been, up to this time, no organisation and no medium of communication among the teachers, except the *American Mathematical Monthly*, which for the past three years has been devoted to this cause. The new organisation, which has been named the Mathematical Association of America, has taken over the *Monthly* as its official journal.

The following officers were elected: President, Professor E. R. Hedrick; First Vice Presidents, Professors E. V. Huntington and G. A. Miller; Secretary-Treasurer, Professor W. D. Cairns; Publication Committee, Professor H. E. Slaught; Managing Editors, Professors W. H. Bussey and R. D. Carmichael.

These officers, together with the following, constitute the Executive Council: Professors R. C. Archibald, Florian Cajori, B. F. Finkel, D. N. Lehmer, E. H. Moore, R. E. Moritz, M. B. Porter, K. D. Swartzel, J. N. Van der Vries, Oswald Veblen, J. W. Young, and Alexander Ziwet.

NOTICE.

By an oversight the name of the King's School, Worcester, did not reach the Hon. Sec. of the Public Schools Special Committee in time to be included in the list of Schools given in No. 120, p. 166.

THE LIBRARY.

Change of Address.

The Library is now at 9 Brunswick Square, W.C., the new premises of the Teachers' Guild.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Scarce Back Numbers.

Reserves are kept of A.I.G.T. Reports and Gazettes, and, from time to time, orders come for sets of these. We are now unable to fulfil such orders for want of certain back numbers, which the Librarian will be glad to buy from any member who can spare them, or to exchange other back numbers for them:

Gazette No. 8 (very important).
A.I.G.T. Report No. 11 (very important).
A.I.G.T. Reports, Nos. 10, 12.

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